Generalized domino-parity inequalities for the TSP

William J. Cook
School of Industrial and Systems Engineering
Georgia Institute of Technology

Daniel Espinoza
Department of Industrial Engineering
Universidad de Chile

Marcos Goycoolea
School of Business
Universidad Adolfo Ibáñez

Abstract

We extend the work of Letchford (2000) by introducing a new class of valid inequalities for the traveling salesman problem, called the generalized domino-parity (GDP) constraints. Just as Letchford's domino-parity constraints generalize comb inequalities, GDP constraints generalize the most well-known multiple-handle constraints, including clique-tree, bipartition, path, and star inequalities. Furthermore, we show that a subset of GDP constraints containing all of the clique-tree inequalities can be separated in polynomial time, provided that the support graph G^* is planar, and provided that we bound the number of handles by a fixed constant h.

1 Introduction

Let $G = (V, E)$ be a complete graph with edge costs $(c_e : e \in E)$. The symmetric traveling salesman problem, or TSP, is to find a minimum-cost hamiltonian circuit in G. In other words, if we identify each node in V with a "city", and each edge cost c_e with the "distance" or "cost" associated to traveling between a pair of cities, the TSP is to find a minimum-cost tour by which to visit every city in G exactly once, and return to the starting point.

In the Dantzig et al. [8] cutting-plane method for the TSP, a tour is represented by 0-1 variables x_e indicating if edge e is to be used in the tour or not. Given a system of linear inequalities $Ax \leq b$ that is satisfied by every tour vector $x = (x_e : e \in E)$, the solution of the linear programming (LP) problem

$$\text{minimize} \sum_{e \in E} c_e x_e \text{ subject to } Ax \leq b$$

provides a lower bound for the TSP. To improve this bound, the cutting-plane method iteratively adds further linear inequalities, or cutting planes, that are satisfied by all tour vectors but not satisfied by the current LP solution vector $x^\star$. This approach is currently the most successful exact solution procedure for solving the TSP; surveys of TSP cutting-plane work can be found in Applegate et al. [2], Jünger et al. [15], and Naddef [19].

Rather than trying to identify just any linear inequality violated by the LP relaxation, the method of Dantzig et al. focuses on identifying specific classes of inequalities; the task of finding a violated inequality among a specified class is known as the *separation problem* for the class. In computational studies, separation schemes are combined with a branch-and-bound search in order to effectively finish solving the problem.

For any $S \subseteq V$, let $\delta(S)$ denote the set of edges with exactly one end in S and let $E(S)$ denote the set of edges having both ends in S. For disjoint sets $S, T \subseteq V$, let $E(S:T)$ denote the set of edges having one end in S and one end in T. For any set $F \subseteq E$, define $x(F) := \sum(x_e : e \in F)$.

A first class of TSP inequalities are the *subtour constraints*

$$x(\delta(S)) \geq 2 \qquad \forall\, \emptyset \neq S \subsetneq V. \tag{1}$$

An important property of these inequalities is that the corresponding separation problem can be solved efficiently, that is, given a non-negative vector x^* a violated constraint can be found in polynomial time, provided one exists.

A second class of TSP inequalities are the *comb constraints*, introduced by Chvátal [6], and Grötschel and Padberg [13]. Given sets $H, T^1, \ldots, T^t \subset V$ such that t is odd, $T^1, \ldots, T^t$ are pairwise disjoint, and for each $i = 1, \ldots, t$ we have $H \cap T^i \neq \emptyset$ and $T^i \setminus H \neq \emptyset$, every tour satisfies the comb constraint

$$x(\delta(H)) + \sum \left(x(\delta(T^i)) : i = 1, \ldots, t \right) \geq 3t + 1.$$

The set H is called the *handle* of the comb and the sets T^i are the *teeth*. Comb constraints are an important component of modern TSP codes, but unlike subtour constraints, no polynomial-time separation algorithm is known.

Many further classes of inequalities have also been studied, extending combs in various ways. For the most part, however, polynomial-time separation algorithms have again proven to be elusive. Although effective heuristic separation algorithms have been developed for various TSP inequalities, additional exact methods could be crucial in pushing TSP codes on to larger test instances.

Fleischer and Tardos [9] began an important study in this context, introducing the use of planar duality in TSP separation algorithms. Given an LP solution vector x^*, the support graph G^* is the subgraph of G induced by the edge set $E^* = \{e \in E : x^*_e > 0\}$. If x^* satisfies all subtour constraints, then a comb inequality can be violated by at most 1.0. Fleischer and Tardos [9] show that if G^* is planar, then a comb inequality violated by 1.0 can be found in polynomial time, provided such a comb exists.

Building on the ideas of Fleischer and Tardos, Letchford [17] introduced a super-class of comb inequalities, called domino-parity constraints, and provided a separation algorithm in the case where G^* is a planar graph. Computational studies by Boyd et al. [4] and Cook et al. [7] have demonstrated the effectiveness of Letchford's method in solving general TSP instances.

In this article we present a generalization of Letchford's results. We begin in Section 2 by describing in detail the domino-parity inequalities and the clique-tree, bipartition, and star classes of multiple-handle extensions of combs. We proceed, in Section 3, to define a new class of inequalities for the TSP that we call the generalized domino-parity (GDP) constraints, generalizing all the afore-mentioned inequalities. In Section 4 we prove that violated GDP constraints may be characterized much in the same way as Letchford [17] characterizes violated domino-parity constraints. In Section 5 we use this characterization to give a polynomial time algorithm which, for any fixed number of handles h, separates a super-class of clique-tree inequalities.

2 Classes of TSP inequalities

We describe several well-known generalizations of comb inequalities.

2.1 The domino-parity inequalities

A *domino* is a pair $\{T_1, T\}$ such that $\emptyset \subsetneq T_1 \subsetneq T \subsetneq V$. Let r be a positive integer and suppose that $E_1, \ldots, E_r \subseteq E$. For each $e \in E$, define $\mu_e = |\{j \in \{1, \ldots, r\} : e \in E_j\}|$. That is, μ_e denotes the number of edge sets in which e appears. Let $H \subsetneq V$. Sets $E_1, \ldots, E_r$ are said to *support the cut* $\delta(H)$ if $\delta(H) = \{e \in E : \mu_e \text{ is odd}\}$. Observe that if $E_1, \ldots, E_r$ supports the cut $\delta(H)$ and x corresponds to a tour, then $\sum_{e\in E} \mu_e x_e$ is even valued. In fact, $\sum_{e\in E} \mu_e x_e = x(\delta(H)) + \sum_{e\in\delta(H)} x_e(\mu_e - 1) + \sum_{e\in E\setminus\delta(H)} \mu_e x_e$, and every term on the right is even-valued. Further, consider a node-set H and the edge-sets $E_1, \ldots, E_r$. There exists a unique edge-set F such that $\{F, E_1, \ldots, E_r\}$ support the cut $\delta(H)$.

Let p be a positive odd integer, and consider $\mathcal{T}$, a collection of p dominoes. Let $H \subseteq V$. Suppose that $F \subseteq E$, together with the sets $\{E(T_1 : T \setminus T_1)\}_{\{T_1,T\}\in\mathcal{T}}$, supports the cut $\delta(H)$ and define μ_e^H accordingly. Letchford [17] showed that every TSP tour satisfies the *domino-parity inequality*

$$\sum_{e\in E} \mu_e^H x_e + \sum_{\{T_1,T\}\in\mathcal{T}} x(\delta(T)) \geq 3p + 1. \tag{2}$$

It is easy to see that domino-parity inequalities generalize comb inequalities. In fact, let $\mathcal{T}$ define the teeth of a comb, and let H be its handle. For every $T \in \mathcal{T}$ define a domino $\{T \cap H, T\}$. These dominoes, together with H, define a domino-parity inequality.

2.2 Bipartition and clique-tree inequalities

Consider families $\mathcal{H} = \{H^1, \ldots, H^h\}$ and $\mathcal{T} = \{T^1, \ldots, T^t\}$, where, $\emptyset \subsetneq H^i \subsetneq V$ for $i = 1, \ldots, h$ and $\emptyset \subsetneq T^j \subsetneq V$ for $j = 1, \ldots, t$. Assume that,

1. $H^i \cap H^j = \emptyset$ for $1 \leq i < j \leq h$,
2. $T^i \cap T^j = \emptyset$ for $1 \leq i < j \leq t$,
3. $T^j \setminus H^i \neq \emptyset$ for $1 \leq i \leq h, 1 \leq j \leq t$.

The sets $H \in \mathcal{H}$ are called handles and the sets $T \in \mathcal{T}$ are called teeth.

For every $j = 1, \ldots, t$ define $t_j = |\{i \in 1, \ldots, h : T^j \cap H^i \neq \emptyset\}|$, and assume $t_j \geq 1$. If $T^j \setminus \bigcup\{H^i : i = 1, \ldots, h\}$ is non-empty, define $\beta_j = 1$, else define $\beta_j = t_j/(t_j - 1)$. For every

$i = 1, \ldots, h$ define $h_i = |\{j \in 1, \ldots, t : H^i \cap T^j \neq \emptyset\}|$, and assume h_i is odd. Boyd and Cunningham [3] proved that the *bipartition inequality*

$$\sum_{i=1}^{h} x(\delta(H^i)) + \sum_{j=1}^{t} \beta_j x(\delta(T^j)) \geq h + \sum_{i=1}^{h} h_i + 2 \sum_{j=1}^{t} \beta_j. \quad (3)$$

is valid for the TSP.

Define a graph whose node-set is the union of $\mathcal{H}$ and $\mathcal{T}$. Define an edge between H^i and T^j in this graph if $H^i \cap T^j \neq \emptyset$. This graph is called the *intersection graph* defined by $\mathcal{H}$ and $\mathcal{T}$. Note that an intersection graph is always bipartite. If every tooth in $T^j \in \mathcal{T}$ is such that $\beta_j = 1$, and in addition, the intersection graph defined by $\mathcal{H}$ and $\mathcal{T}$ is a tree, it is easy to see that (3) is equivalent to,

$$\sum_{i=1}^{h} x(\delta(H^i)) + \sum_{j=1}^{t} x(\delta(T^j)) \geq 2h + 3t - 1. \quad (4)$$

This constraint, introduced by Grötschel and Pulleyblank [14], is known as a *clique-tree inequality.* A clique-tree inequality having a single handle is a comb inequality.

In Figure 1 we illustrate a bipartition constraint with three handles and ten teeth having $\beta_j = 1$. This constraint has the form

$$\sum_{i=1}^{3} x(\delta(H_i)) + \sum_{j=1}^{10} x(\delta(T_j)) \geq 44.$$

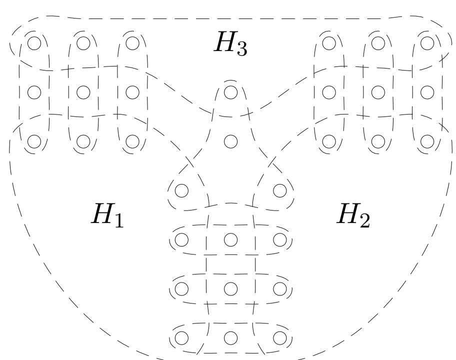

Figure 1: Example of a bipartition constraint on three handles.

2.3 Star and path inequalities

As before, consider the collections $\mathcal{H} = \{H^1, \ldots, H^h\}$ and $\mathcal{T} = \{T^1, \ldots, T^t\}$, where, $\emptyset \subsetneq H^i \subsetneq V$ for $i = 1, \ldots, h$ and $\emptyset \subsetneq T^j \subsetneq V$ for $j = 1, \ldots, t$. Now, assume

1. $H^1 \subset H^2 \subset \ldots \subset H^h$,
2. $T^j \cap T^k = \emptyset \qquad \forall 1 \leq j < k \leq t$,
3. $H^1 \cap T^j \neq \emptyset$ for $1 \leq j \leq t$,

4. $T^j \setminus H^h \neq \emptyset$ for $1 \leq j \leq t$,
5. $(H^{i+1} \setminus H^i) \setminus \bigcup_{j=1}^t T^j = \emptyset$ for $1 \leq i \leq h-1$.

The set $\hat{I} = \{l, l+1, \ldots, l+r\} \subseteq \{1, \ldots, h\}$ is an *interval* corresponding to tooth T_j if $H^i \cap T^j = H^k \cap T^j$ for all $i, k \in \hat{I}$ and if $H^i \cap T^j \neq H^k \cap T^j$ for all $i \in \hat{I},\ k \notin \hat{I}$, that is, $\hat{I}$ is a maximal index set of (successive) handles which have the same intersection with T^j. Consider $\alpha \in \mathbb{N}^h$, and $\gamma \in \mathbb{N}^t$. Assume $\mathcal{H}$ and $\mathcal{T}$ satisfy the interval property with regards to α and γ, that is, assume for each $1 \leq j \leq t$, and each interval $\hat{I}$ of T^j, we have $\gamma_j \geq \sum_{i \in \hat{I}} \alpha_i$. Fleischmann [10] showed that the following *star inequality*

$$\sum_{i=1}^{h} \alpha_i x(\delta(H^i)) + \sum_{j=1}^{t} \gamma_j x(\delta(T^j)) \geq (t+1) \sum_{i=1}^{h} \alpha_i + 2 \sum_{j=1}^{t} \gamma_j \tag{5}$$

is valid for the TSP. An important special case of stars are the *path (PWB) inequalities* studied by Naddef and Rinaldi [20], again generalizing combs to multiple handles.

3 Generalized domino-parity (GDP) inequalities

In this section, we construct a generalization of the domino-parity inequalities and show that this generalized class of constraints strictly contains bipartition and star inequalities.

3.1 Multi-dominoes

For $U \subseteq V$ define $G[U]$ as the complete sub-graph of G induced by U. Consider a non-negative integer k and a family of node sets $\hat{T} = \{T_1, T_2, \ldots, T_k, T\}$ such that $\emptyset \neq T_i \subsetneq T \subsetneq V,\ \forall i = 1, \ldots, k$. We say that this family defines a *multi-domino* if for any set $\emptyset \neq K \subseteq \{1, \ldots, k\}$, the edges $\bigcup \{E(T_i : T \setminus T_i) : i \in K\}$ define a $|K| + 1$ (or greater) cut in $G[T]$.

Consider a positive integer k and a family of node sets $\hat{T} = \{T_1, T_2, \ldots, T_k, T\}$ satisfying $\emptyset \neq T_i \subsetneq T \subsetneq V,\ \forall i = 1, \ldots, k$. $\hat{T}$ is said to define a *degenerate multi-domino* if $\{T_1, \ldots, T_k\}$ defines a partition of T.

Note that unless otherwise specified, we will use the term multi-domino to refer both to degenerate and non-degenerate multi-dominoes.

In general, given a multi-domino $\hat{T} = \{T_1, \ldots, T_k, T\}$, we will say that T is its *ground-set*, and $T_1, \ldots, T_k$ are its *halves*. If a multi-domino $\hat{T}$ has k halves, we say that it is a *k-domino*, and write $\kappa(\hat{T}) = k$. Observe that k-dominoes (both degenerate and non-degenerate) satisfy the following recursive condition: If you remove any number $0 < r \leq k$ of halves from a k-domino (leaving the ground set intact), you obtain a $k - r$ domino which is non-degenerate. In addition, note that the definition of a 1-domino is equivalent to the domino definition of Letchford [17], and a 0-domino consists of a singleton containing a ground set and no halves. Whenever a multi-domino has more than one half, we will say that it is *large*. Finally, observe that for notation purposes, we will distinguish a multi-domino $\hat{T}$ from its ground set T by using a hat (“ ˆ ”) symbol.

If a k-domino $\hat{T}$ is degenerate, define $\beta(\hat{T}) = \frac{k}{k-1}$. Otherwise, define $\beta(\hat{T}) = 1$.

Lemma 1. *Let $\hat{T} = \{T_1, T_2, \ldots, T_k, T\}$ be a $k-$domino. If x satisfies all subtour constraints, then*

$$\frac{\beta(\hat{T})}{2}(x(\delta(T)) - 2) + \sum_{i=1}^{k} x(E(T_i : T \setminus T_i)) \geq k.$$

Proof: Assume x satisfies all subtour constraints. If $k = 0$ the result trivially follows from the subtour constraints, so assume $k \geq 1$ and let $B_1, B_2, \ldots, B_r$ correspond to the partition of T obtained by removing the edge sets $E(T_1 : T \setminus T_1), E(T_2 : T \setminus T_2), \ldots, E(T_k : T \setminus T_k)$ from $G[T]$. Note that

$$\sum_{i=1}^{r} x(\delta(B_i)) = x(\delta(T)) + \sum_{i=1}^{r} x(E(B_i : T \setminus B_i)).$$

It follows that

$$\frac{\beta(\hat{T})}{2}(x(\delta(T)) - 2) = \frac{\beta(\hat{T})}{2}\left(\sum_{i=1}^{r}(x(\delta(B_i)) - x(E(B_i : T \setminus B_i))) - 2\right). \tag{6}$$

However, note that if $\hat{T}$ is non-degenerate, then $\beta(\hat{T}) = 1$ and

$$\sum_{i=1}^{r} x(E(B_i : T \setminus B_i)) \leq 2\sum_{i=1}^{k} x(E(T_i : T \setminus T_i)).$$

On the other hand, if $\hat{T}$ is degenerate, then $\beta(\hat{T}) \leq 2$ and each T_i can be assumed equal to B_i. Thus, in either case, we have

$$\frac{\beta(\hat{T})}{2}\sum_{i=1}^{r} x(E(B_i : T \setminus B_i)) \leq \sum_{i=1}^{k} x(E(T_i : T \setminus T_i)). \tag{7}$$

Finally, note that if $\hat{T}$ is non-degenerate, then $r > k$ and $\beta(\hat{T}) = 1$. Likewise, if $\hat{T}$ is degenerate then $r = k$ and $\beta(\hat{T}) = k/(k-1)$. Thus, in both cases, $\beta(\hat{T})(2r-2)/2 \geq k$, and

$$\frac{\beta(\hat{T})}{2}\left(\sum_{i=1}^{r} x(\delta(B_i)) - 2\right) \geq \frac{\beta(\hat{T})}{2}(2r - 2) \geq k. \tag{8}$$

Putting together (6), (7), and (8) we get the desired result. □

3.2 Defining the GDP inequalities

Recall that comb inequalities require an odd number of teeth to intersect the handle of the constraint. However, for domino-parity inequalities, this requirement is relaxed, and though no conditions are imposed in terms of intersections, an odd number of teeth is still associated to the handle, but in a more abstract way through the notion of "supporting a cut". Again, in bipartition inequalities, handles are required to intersect an odd number of teeth. Since we are interested in generalizing domino-parity constraints to a class of inequalities containing bipartitions, we will need to generalize this association between handles and teeth to multiple handle configurations. In order to do this, we will map teeth, which in our new inequalities will be represented by multi-dominoes, to handles, by means of a function Φ, which associates each half of a multi-domino to a handle.

Consider a positive integer h and a family $\mathcal{T}$ of multi-dominoes. Let Φ define a map between halves of the multi-dominoes in $\mathcal{T}$ and numbers in $\{1,\dots,h\}$. That is, for every multi-domino $\hat{T} \in \mathcal{T}$, such that $\kappa(\hat{T}) \geq 1$, and every $j \in 1,\dots,\kappa(\hat{T})$, let $\Phi(\hat{T},j)$ take a value in $\{1,\dots,h\}$. We say that Φ is an h-tooth association defined over $\mathcal{T}$, and whenever $\Phi(\hat{T},j) = i$ for some $i \in \{1,\dots,h\}$ we will say that the i-th handle and the j-th half of $\hat{T}$ are associated to each other by means of Φ.

Theorem 2. *Consider a family of node sets $\mathcal{H} = \{H_1,\dots,H_h\}$, and a family of multi-dominoes $\mathcal{T}$. Let Φ define an h-tooth association over $\mathcal{T}$, and assume that $|\Phi^{-1}(i)|$ is odd, for each $i = 1,\dots,h$. For each $H_i \in \mathcal{H}$ define $F_i \subseteq E$ such that $\{F_i\}$ and $\{E(T_j : T \setminus T_j) : \Phi(\hat{T},j) = i\}$ support the cut $\delta(H_i)$ in G and define μ^i accordingly. Then the inequality*

$$\sum_{i=1}^{h} \mu^i x + \sum_{\hat{T}\in\mathcal{T}} \beta(\hat{T}) x(\delta(T)) \geq h + \sum_{i=1}^{h} h_i + 2\sum_{T\in\mathcal{T}} \beta(\hat{T}) \tag{9}$$

is satisfied by all tours, where $h_i = |\Phi^{-1}(i)|$ for each $i = 1,\dots,h$.

Proof: We use induction on h, the case $h = 0$ following from the validity of the subtour constraints. Let $\hat{x}$ be the incidence vector of a tour. If there exists $i_o \in \{1,\dots,k\}$ such that $\mu^{i_o}\hat{x} > h_{i_o} - 1$, then, since $\mu^{i_o}\hat{x}$ is even valued (see Section 2.1), we have $\mu^{i_o}\hat{x} \geq h_{i_o} + 1$. For every $\hat{T} \in \mathcal{T}$ define $\hat{T}^* = \{T_1, T_2, \dots, T_{\kappa(T)}, T\} \setminus \{T_j : \Phi(\hat{T},j) = i_o\}$. Note that for each $\hat{T} \in \mathcal{T}$ we have $\beta(\hat{T}^*) \leq \beta(\hat{T})$. Thus, by induction, the inequality obtained by removing handle H_{i_o}, replacing each $\hat{T} \in \mathcal{T}$ by $\hat{T}^*$, using the same association Φ, and renumbering appropriately,

$$\sum_{i=1, i\neq i_o}^{k} \mu^i x + \sum_{\hat{T}^*\in\mathcal{T}} \beta(\hat{T}^*) x(\delta(T)) \geq (h-1) + \sum_{i=1, i\neq i_o}^{k} h_i + 2\sum_{\hat{T}^*\in\mathcal{T}} \beta(\hat{T}^*)$$

is valid. Then (9) follows since $(\beta(\hat{T}) - \beta(\hat{T}^*))\hat{x}(\delta(T)) \geq (\beta(\hat{T}) - \beta(\hat{T}^*))2$, and $\mu^{i_o}\hat{x} \geq h_{i_o} + 1$. Now assume $\mu^i\hat{x} \leq h_i - 1$ for each $i = 1,\dots,h$. From Lemma 1 we have for each $T \in \mathcal{T}$

$$\beta(\hat{T})(\hat{x}(\delta(T)) - 2) \geq 2\kappa(T) - 2\sum_{j=1}^{\kappa(T)} \hat{x}(E(T_j : T \setminus T_j)). \tag{10}$$

Noting that,

$$\sum_{\hat{T}\in\mathcal{T}} \kappa(\hat{T}) = \sum_{i=1}^{h} h_i,$$

and,

$$\sum_{i=1}^{h} \hat{x}(F_i) + \sum_{\hat{T}\in\mathcal{T}} \sum_{j=1}^{\kappa(\hat{T})} \hat{x}(E(T_j : T \setminus T_j)) = \sum_{i=1}^{h} \mu^i \hat{x}$$

and then summing over (10) we obtain,

$$\begin{aligned}\sum_{T\in\mathcal{T}} \beta(\hat{T})(\hat{x}(\delta(T))-2) &\geq 2\sum_{\hat{T}\in\mathcal{T}} \kappa(\hat{T}) - 2\sum_{\hat{T}\in\mathcal{T}}\sum_{j=1}^{\kappa(\hat{T})} \hat{x}(E(T_j : T\setminus T_j)) \\ &\geq 2\sum_{i=1}^{h} h_i - 2\sum_{i=1}^{h}\mu^i\hat{x} \\ &= \sum_{i=1}^{h} h_i + \sum_{i=1}^{h}\left(h_i - \mu^i\hat{x}\right) - \sum_{i=1}^{h}\mu^i\hat{x} \\ &\geq \sum_{i=1}^{h} h_i + h - \sum_{i=1} \mu^i\hat{x}.\end{aligned}$$

□

We refer to the inequalities (9) as *generalized domino-parity (GDP) inequalities.* As in other well-known TSP inequalities, we will denote the sets $H_1, \ldots, H_h$ as *handles*, and the multi-dominoes $\hat{T} \in \mathcal{T}$ as *teeth*. We will say that teeth with more than one half are *large teeth*. If a multi-parity constraint has h handles, we say that it is an *h-parity constraint.* When $h = 1$, and every tooth is non-degenerate and restricted to having at most one half, this class coincides with that of the *domino-parity inequalities* of Letchford [17]. In order to represent generalized domino-parity inequalities we will identify them in terms of the tuples $(\mathcal{H}, \Phi, \mathcal{T})$, or equivalently, the tuples $(\mathcal{F}, \Phi, \mathcal{T})$, corresponding to the handles (or sets F_i), the h-tooth association, and the teeth which define them. Note that handles in a GDP inequality can be empty.

3.3 Star and bipartition inequalities are GDP inequalities

Proposition 3. *The class of $h-$parity inequalities contains the class of bipartition inequalities having h handles.*

Proof: Consider a bipartition inequality with handles $\mathcal{H} = \{H^1, \ldots, H^h\}$ and teeth $\mathcal{T} = \{T^1, \ldots, T^t\}$. Define a set of multi-dominoes $\mathcal{T}'$ and an h-tooth association Φ by repeating the following procedure.

- Step 1. Choose a tooth $T^j \in \mathcal{T}$ and define a zero-domino $\hat{T}^j \in \mathcal{T}'$ having ground set T^j.
- Step 2. For each handle $H^i \in \mathcal{H}$ such that $T^j \cap H^i \neq \emptyset$: Let $r = \kappa(\hat{T}^j)$. Add half $T^j \cap H^i$ to $\hat{T}$, and define $\Phi(\hat{T}, r+1) = i$.

It is easy to see that this procedure leads to a valid h-parity inequality which coincides with the original bipartition inequality. □

Given that bipartition inequalities generalize clique-tree inequalities, Proposition 3 also tells us that h-parity inequalities generalize clique-tree inequalities on h handles.

A more involved argument shows that star inequalities are GDP inequalities; details of the construction can be found in the Ph.D. thesis of Goycoolea [12].

Proposition 4. *The class of GDP inequalities contains the class of star inequalities.* □

The construction in Goycoolea [12] shows that star inequalities are GDP inequalities such that: (a) for every pair of handles, one must contain the other, and (b) every pair of teeth either have completely disjoint ground sets, or the ground sets are exactly alike.

Note that in addition to containing bipartition and star inequalities, the class of GDP inequalities contains many other new and different structures. It is easy to see that not all GDP inequalities define facets of the TSP polytope, but the class does provide a common framework for possibly extending Letchford's algorithm to super-classes of other inequalities that have proven to be effective in TSP codes.

4 Properties of violated GDP inequalities

In this section we describe necessary and sufficient conditions for an h-parity constraint to be violated.

4.1 A characterization of violated GDP inequalities

Define the *weight* of a k-domino $\hat{T} = \{T_1, T_2, \ldots, T_k, T\}$ to be

$$w(\hat{T}) := \beta(\hat{T})(x(\delta(T)) - 2) + \sum_{i=1}^{k} x(E(T_i : T \setminus T_i)) - k. \tag{11}$$

Lemma 5. *Consider an h-parity inequality defined by $\mathcal{H}$, $\mathcal{T}$, and Φ. Let F_i be such that the edge sets $\{E(T_j : T \setminus T_j) : \Phi(\hat{T}, j) = i\}$ and $\{F_i\}$ support the cut $\delta(H_i)$ for each $i = 1, \ldots, h$. The slack of the h-parity inequality is*

$$\sum_{\hat{T} \in \mathcal{T}} w(\hat{T}) + \sum_{i=1}^{h} x(F_i) - h.$$

Proof: The slack is

$$\begin{aligned}
& \sum_{i=1}^{h} \mu^i x + \sum_{\hat{T} \in \mathcal{T}} \beta(\hat{T}) x(\delta(T)) - h - \sum_{i=1}^{h} h_i - 2 \sum_{\hat{T} \in \mathcal{T}} \beta(\hat{T}) \\
= \; & \sum_{i=1}^{h} x(F_i) + \sum_{\hat{T} \in \mathcal{T}} \sum_{j=1}^{\kappa(\hat{T})} x(E(T_j : T \setminus T_j)) + \sum_{\hat{T} \in \mathcal{T}} \left(\beta(\hat{T}) \left(x(\delta(T)) - 2 \right) - \kappa(\hat{T}) \right) - h \\
= \; & \sum_{i=1}^{h} x(F_i) + \sum_{\hat{T} \in \mathcal{T}} \left(\beta(\hat{T}) \left(x(\delta(T)) - 2 \right) + \sum_{j=1}^{\kappa(\hat{T})} x(E(T_j : T \setminus T_j)) - \kappa(\hat{T}) \right) - h \\
= \; & \sum_{i=1}^{h} x(F_i) + \sum_{\hat{T} \in \mathcal{T}} w(\hat{T}) - h.
\end{aligned}$$

□

Note that Lemma 1 and Lemma 5 together imply that if x satisfies all subtour constraints, then a violated h-parity constraint must satisfy

$$0 \leq \frac{\beta(\hat{T})}{2}(x(\delta(T)) - 2) \leq w(\hat{T}) < h \qquad \forall \hat{T} \in \mathcal{T}. \tag{12}$$

Further, note that in many classes of well-known TSP inequalities, the handles are disjoint and halves of a multi-domino correspond to tooth-handle intersections (for example, clique-tree inequalities and bipartition inequalities—see Proposition 3). In these cases it is not difficult to see that every k-domino $\hat{T}$ participating in such inequalities will satisfy $w(\hat{T}) \geq \sum_{j=1}^{k} x(\delta(T_j)) - k - 2$. Thus, if all subtour constraints are satisfied, $w(\hat{T}) \geq k - 2$. This means that it is possible to bound the number of teeth having three or more halves which participate in violated h-parity constraints with such a characteristic.

Let x^* be an LP solution vector, and define the corresponding support graph G^*. We now describe several characteristics of violated h-parity inequalities.

Lemma 6. *There exists an h-parity inequality with slack s^* if and only if there exist a family of multi-dominoes $\mathcal{T}$, an h-tooth association Φ, and sets $R_i \subseteq E^*$ for all $i \in \{1, \ldots, h\}$ such that:*

1. $|\Phi^{-1}(i)|$ *is odd, for all* $i = 1, \ldots, h$.
2. $\{E^*(T_j : T \setminus T_j) : \Phi(\hat{T}, j) = i\}$ *and* $\{R_i\}$ *support a cut in* G^*, *for all* $i = 1, \ldots, h$.
3. $s^* = \sum_{i=1}^{h} x^*(R_i) + \sum_{\hat{T} \in \mathcal{T}} w(\hat{T}) - h$.

Proof: From Theorem 2 and Lemma 5, there exists an h-parity inequality with slack s^* if and only if there exists a family of node sets $\mathcal{H} = \{H_1, \ldots, H_h\}$ in G, a family of edge sets $\{F_1, \ldots, F_h\}$ in G, a family of multi-dominoes $\mathcal{T}$ in G, and an h-tooth association Φ defined over $\mathcal{T}$, such that:

(a) $|\Phi^{-1}(i)|$ is odd, for $i = 1, \ldots, h$.
(b) $\{E(T_j : T \setminus T_j) : \Phi(\hat{T}, j) = i\}$ and $\{F_i\}$ support the cut $\delta(H_i)$ in G for all $i = 1, \ldots, h$.
(c) $s^* = \sum_{i=1}^{h} x^*(F_i) + \sum_{\hat{T} \in \mathcal{T}} w(\hat{T}) - h$.

Necessity is trivial, since we can just take $R_i = F_i \cap E^*$. So we focus on sufficiency. Assume that $\mathcal{T}$ and Φ define an h-tooth association, and sets $R_i \subseteq E^*$ for $i \in 1, \ldots, h$ are such that (1) and (2) hold. For each $i \in 1, \ldots, h$ let $H_i \subseteq V$ be one shore of the cut supported by $\{E^*(T_j : T \setminus T_j) : \Phi(\hat{T}, j) = i\}$ and R_i. A set F_i satisfying (b) and (c) can be obtained from R_i by adding edges $e \in \delta(H_i)$ such that $x_e^* = 0$. □

We say that a one-domino $\{T_1, T\}$ is super-connected in G^* if the cuts $\delta^*(T_1), \delta^*(T)$, and $\delta^*(T \setminus T_1)$ are all minimal. We say that a one-domino is trivial it is of the form $\{\{u\}, \{u, v\}\}$ with $u, v \in V$. The following Lemma, which generalizes a result of Letchford [17] will be key in the separation algorithm we will later develop.

Lemma 7. *Consider a fractional solution x^*. There exists a maximally violated h-parity constraint $(\mathcal{F}, \Phi, \mathcal{T})$ such that every 1-domino $\hat{T} = \{T_1, T\} \in \mathcal{T}$ is super-connected.*

The proof of this Lemma follows directly from the following results, the first of which admits a straight-forward proof.

Lemma 8. *Let edge sets $\{E_1, \ldots, E_k\}$ support a cut in G.*

(a) Let $S \subseteq E(G)$ define a cut $\delta(A)$ for some $A \subsetneq V$, and assume $S = S_1 \cup S_2 \cup \ldots \cup S_k$, where the sets $S_1, \ldots, S_k$ are pairwise disjoint. Then, $\{E_1 \Delta S_1, \ldots, E_k \Delta S_k\}$ supports a cut in G, though not necessarily the same cut supported by $\{E_1, \ldots, E_k\}$.

(b) Consider any edge set $S \subseteq E(G)$, then $\{E_1 \Delta S, E_2 \Delta S, E_3, \ldots, E_k\}$ supports the same cut as $\{E_1, E_2, \ldots, E_k\}$ in G. □

Lemma 9. *Consider a fractional solution x^* and an h-parity constraint $(\mathcal{F}, \Phi, \mathcal{T})$ such that $\hat{T} = \{T_1, T\} \in \mathcal{T}$ is not super-connected. It is possible to replace $\hat{T}$ with a trivial one-domino and obtain another h-parity constraint with less than or equal slack than that of $(\mathcal{F}, \Phi, \mathcal{T})$.*

Proof: Consider any edge $e = \{u, v\} \in E$ and define a trivial one-domino $\hat{T}' = \{\{u\}, \{u, v\}\}$. Observe that $w(\hat{T}') = 1 - x^*_e$. Assume that $\Phi(\hat{T}, 1) = i$ and let $\{F_i, E(T_1 : T \backslash T_1), E_1, \ldots, E_k\}$ be the support set of the i-th handle, where sets $E_1, \ldots, E_k$ correspond to the edge sets derived from other multi-dominoes.

Case 1: Assume that $\delta(T)$ is not a minimal cut. In this case, we have that $x^*(\delta(T)) \geq 4$. From part (b) of Lemma 8, we know that $\{F_i, E(T_1 : T \setminus T_1), E_1, \ldots, E_k\}$ and $\{(F_i \Delta E(T_1 : T \backslash T_1)) \Delta \{e\}, \{e\}, E_1, \ldots, E_k\}$ support the same handle. Let $F'_i = (F_i \Delta E(T_1 : T \backslash T_1)) \Delta \{e\}$. Observe that,

$$
\begin{aligned}
x^*(F'_i) + w(\hat{T}') &\leq (x(F_i) + x(E(T_1 : T \setminus T_1)) + x_e) + (1 - x_e) \\
&= x^*(F_i) + 1 + x(E(T_1 : T \setminus T_1)) \\
&\leq x^*(F_i) + (x(\delta(T)) - 3) + x(E(T_1 : T \setminus T_1)) \\
&= x^*(F_i) + x(w(T))
\end{aligned}
$$

Let $\mathcal{F}' = \mathcal{F} \setminus \{F_i\} \cup \{F'_i\}$ and $\mathcal{T}' = \mathcal{T} \setminus \{\hat{T}\} \cup \{\hat{T}'\}$. Further, define Φ' equal to Φ but for the fact that $\Phi'(\hat{T}', 1) = i$. From Lemma 6 we know that $(\mathcal{F}', \Phi', \mathcal{T}')$ defines a valid h-parity inequality, and from Lemma 5 it follows that the slack of this new inequality is less than or equal that of $(\mathcal{F}, \Phi, \mathcal{T})$. Thus we conclude our result.

Case 2: Assume that $\delta(T_1)$ is not a minimal cut. In this case, we have that $x^*(\delta(T_1)) \geq 4$. Observe that $E(T \setminus T_1 : T^c)$ and $E(T_1 : T \setminus T_1)$ are disjoint, and that $\delta(T \setminus T_1) = E(T \setminus T_1 : T^c) \cup E(T_1 : T \setminus T_1)$. Applying part (a) with $S_1 = E(T \setminus T_1 : T^c)$ and $S_2 = E(T_1 : T \setminus T_1)$ and then part (b) with $S = \{e\}$ of Lemma 8 , we know that because $\{F_i, E(T_1 : T \setminus T_1), E_1, \ldots, E_k\}$ supports a cut, then so does $\{(F_i \Delta E(T \setminus T_1 : T^c)) \Delta \{e\}, \{e\}, E_1, \ldots, E_k\}$ (though not necessarily the same one). Let $F'_i = (F_i \Delta E(T \setminus T_1 : T^c)) \Delta \{e\}$. Observe that,

$$
\begin{aligned}
x^*(F'_i) + w(\hat{T}') &\leq (x(F_i) + x(E(T \setminus T_1 : T^c)) + x_e) + (1 - x_e) \\
&= x^*(F_i) + 1 + x(E(T \setminus T_1 : T^c)) \\
&\leq x^*(F_i) + (x(\delta(T_1)) - 3) + x(E(T \setminus T_1 : T^c)) \\
&= x^*(F_i) + x(w(T))
\end{aligned}
$$

Let $\mathcal{F}' = \mathcal{F} \setminus \{F_i\} \cup \{F'_i\}$ and $\mathcal{T}' = \mathcal{T} \setminus \{\hat{T}\} \cup \{\hat{T}'\}$. Further, define Φ' equal to Φ but for the fact that $\Phi'(\hat{T}', 1) = i$. From Lemma 6 we know that $(\mathcal{F}', \Phi', \mathcal{T}')$ defines a valid h-parity inequality, and from Lemma 5 it follows that the slack of this new inequality is less than or equal that of $(\mathcal{F}, \Phi, \mathcal{T})$. Thus we conclude our result.

Case 3: Assume that $\delta(T \setminus T_1)$ is not a minimal cut. In this case, the proof is analogous to that of Case 2. □

4.2 Planar duality and violated GDP inequalities

We henceforth assume that G^* is a planar graph and let $\bar{G^*}$ denote the planar dual of G^*. For any subset $F \subseteq E(G^*)$, denote by $\bar{F}$ the corresponding edges in $\bar{G^*}$. For each $\bar{e} \in \bar{G^*}$ let $x^*_{\bar{e}} = x^*_e$.

A graph is called *eulerian* if every node has even degree. As in Letchford [17], we do not require that eulerian graphs be connected.

Let r be a positive integer and suppose $E_1, \ldots, E_r \subseteq E^*$. As before, let $\mu_e = |\{i : e \in E_i\}|$. The collection $\{\bar{E}_i : i = 1, \ldots, r\}$ is said to *support an eulerian subgraph* in $\bar{G^*}$ if the edges $\bar{e}$ for which μ_e is odd form an eulerian subgraph in $\bar{G^*}$.

A cut $C \subseteq E(G)$ is *minimal* if removing any subset of edges from C results in an edge set which does not define a cut. Observe that for any set $A \subsetneq V$ the cut $\delta(A)$ can always be decomposed into an edge disjoint union of minimal cuts. A well known result (see Mohar and Thomassen [18]) is that if G is planar and $C \subseteq E(G)$ is a minimal cut, then $\overline{C}$ is a simple cycle in $\bar{G^*}$. Since every eulerian subgraph can be decomposed into edge disjoint simple cycles, this result implies that $\{\bar{E}_i : i = 1 \ldots, r\}$ supports an eulerian subgraph in $\bar{G^*}$ if and only if $\{E_i : i = 1, \ldots, r\}$ supports a cut in G^*. This observation implies the following dual version of Lemma 6.

Lemma 10. *There exists an h-parity inequality having slack s^* if and only if there exist a family of multi-dominoes $\mathcal{T}$, an h-tooth association Φ, and sets $\bar{R}_i \subseteq \bar{E}^*$ for all $i \in \{1, \ldots, h\}$ such that:*

1. *$|\Phi^{-1}(i)|$ is odd, for all $i = 1, \ldots, h$.*
2. *$\{\overline{E^*(T_j : T \setminus T_j)} : \Phi(\hat{T}, j) = i\}$ and $\{\bar{R}_i\}$ support an eulerian subgraph in $\bar{G^*}$ for all $i = 1, \ldots, h$.*
3. *$s^* = \sum_{i=1}^{h} x^*(\bar{R}_i) + \sum_{\hat{T} \in \mathcal{T}} w(\hat{T}) - h$.* □

In Figure 2 we illustrate the relevant edges of a two-parity constraint in the dual graph $\bar{G^*}$. This example has a total of six teeth, with one of the teeth strictly containing two other teeth. In the illustration, the dark circles represent nodes in $\bar{V}^*$, the solid lines represent edges in $\bar{R}_i$ for $i = 1, 2$, the dashed lines represent edges in $\overline{\delta(T)}$ for $T \in \mathcal{T}$, and the dotted lines represent edges in $\overline{E(T_j : T \setminus T_j)}$ for $j \in 1, \ldots, h$ such that $\Phi(T, j) \in \{1, 2\}$. Note that the solid and dotted lines together support two eulerian subgraphs, one for each handle.

Planarity also allows us to strengthen our characerization of one-dominoes which participate in violated h-parity inequalities. The following result, by Letchford [17], is very powerful when combined with Lemma 7.

Theorem 11 (Letchford 2000)**.** *Let x^* be a fractional solution satisfying all of the subtour constraints. Let $\hat{T} = \{T_1, T\}$ be a super-connected 1-domino. If G^* is a planar graph, then*

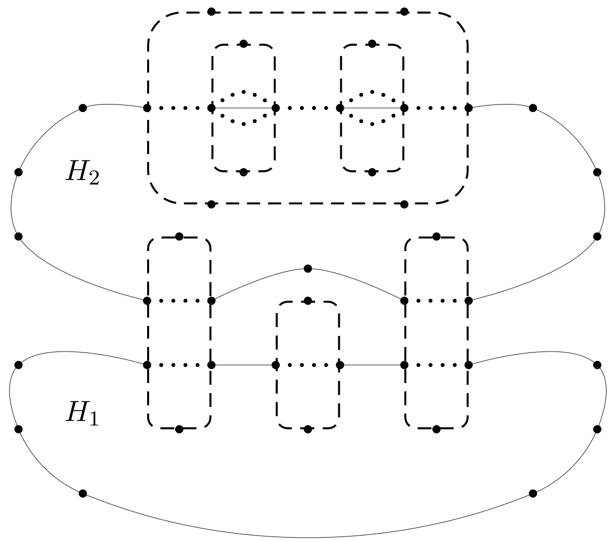

Figure 2: A two-parity constraint represented in $\bar{G}^*$

there exist two vertices $s, t \in \bar{G}^*$ such that each of the following edge sets is an (s,t) path in $\bar{G}^*$:

(i) $\overline{E^*(T_1 : T \setminus T_1)}$,

(ii) $\overline{E^*(T_1 : V \setminus T_1)}$,

(iii) $\overline{E^*(T \setminus T_1 : V \setminus T)}$.

Furthermore, these three (s,t) paths are edge disjoint and also have no vertices in common other than s and t. □

An example of this result is depicted in Figure 3. In illustration (a) a 1-domino is drawn in G^*. In this picture, the dotted lines represent the boundary of the domino and of its half, the dashed lines represent the edges in $\delta(T) \cup E(T_1 : T \setminus T_1)$, and the solid lines represent other edges in the graph. In illustration (b) the edges defining the same domino are depicted in $\bar{G}^*$. Again, the dashed lines correspond to the edges in $\delta(T) \cup E(T_1 : T \setminus T_1)$, and the solid lines correspond to the remaining edges. As can be seen in the latter illustration, the dashed lines define three edge disjoint paths which join vertices s and t, and which do not have any other vertices in common.

5 Separating GDP inequalities in planar graphs

Let $\mathcal{D}(h, l, r)$ represent the set of all generalized domino-parity inequalities $(\mathcal{H}, \Phi, \mathcal{T})$ such that (a) the number of handles is not greater than h, e.g. $|\mathcal{H}| \leq h$, (b) the number of large teeth is not greater than l, e.g. $|\hat{T} \in \mathcal{T} : \kappa(\hat{T}) \geq 2| \leq l$, and (c) no tooth has more than r halves, e.g. $\kappa(\hat{T}) \leq r$ for all $\hat{T} \in \mathcal{T}$.

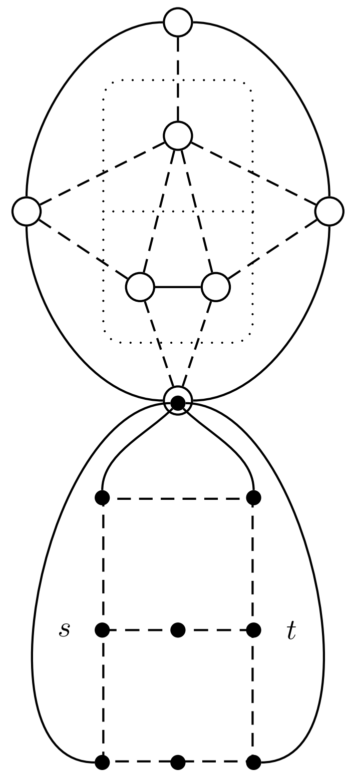

Figure 3: Representation of a domino in G^* and $\bar{G^*}$.

In this section we present a polynomial-time algorithm which, for fixed integers h, l, r, finds a maximally violated constraint in $\mathcal{D}(h, l, r)$ in polynomial time, provided G^* is planar.

The algorithm proceeds in two steps. First, a set of candidate teeth is generated. Second, an enumeration scheme tests different associations Φ with the candidate teeth in order to identify a maximally violated inequality.

5.1 Step 1: Generating a set of candidate teeth

We begin with a simple result which establishes a bound on the weight of teeth participating in violated inequalities.

Lemma 12. *Let x^* be a fractional solution, and let $\hat{T}$ correspond to a k-domino participating in a violated h-parity constraint. Then,*

$$x^*(\delta(T)) + \sum_{j=1}^{k} x^*(E(T_j : T \setminus T_j)) < h + k + 2. \tag{13}$$

Proof: We know that

$$w(\hat{T}) = \beta(\hat{T})(x^*(\delta(T)) - 2) + \sum_{i=1}^{k} x^*(E(T_i : T \setminus T_i)) - k.$$

From equation (12) we know that $w(\hat{T}) < h$. In addition, we know $\beta(\hat{T}) \geq 1$. Thus,

$$(x^*(\delta(T)) - 2) + \sum_{i=1}^{k} x^*(E(T_i : T \setminus T_i)) - k \leq w(\hat{T}) < h.$$

The result immediately follows. □

Consider a fractional solution x^* and non-negative integers h, l, r. We say that a set of multi-dominoes $\mathcal{L}^*$ is *complete* for $\mathcal{D}(h, l, r)$ if every constraint $(\mathcal{H}, \Phi, \mathcal{T}) \in \mathcal{D}(h, l, r)$ which is violated by x^* satisfies $\mathcal{T} \subseteq \mathcal{L}^*$.

The importance of Lemma 12 is that it allows us to construct a complete set of teeth, as indicated by the following proposition.

Proposition 13. *Consider x^* satisfying all of the subtour constraints and non-negative integers h, l, r. It is possible to construct a complete set of multi-dominoes for $\mathcal{D}(h, l, r)$ in $O(n^{r^2+(h+3)(r+1)+1})$.*

Proof: Consider a k-domino $\hat{T}$ (with $k \leq r$) participating in a violated h-parity constraint. Lemma 12 implies $x^*(\delta(T)) < h + k + 2$. In addition, because $\delta(T_j) \subseteq T \cup E(T_j : T \setminus T_j)$ for each $j = 1, \ldots, k$, it also implies that $x^*(\delta(T_j)) < h + k + 2$. Thus, it can be seen that the building blocks the multi-dominoes required are the sets $A \subsetneq V(G^*)$ such that $x^*(\delta(A)) < h + r + 2$.

Let $\mathcal{A} = \{A \subsetneq V : x^*(\delta(A)) < h+r+2\}$. Since all sets $A \subsetneq V(G^*)$ satisfy $x^*(\delta(A)) \geq 2$, we know that each $A \in \mathcal{A}$ is within a factor of $\alpha = (h + r + 2)/2$ of the min-cut. From Karger [16] it follows that $|\mathcal{A}| \leq (2n)^{h+r+2}$. Using the algorithm of Nagamochi [21] it is possible to completely enumerate $\mathcal{A}$ in $O(m^2 n + n^{h+r+2} m)$, where m is the number of edges in G. Since in planar graphs $m = O(n)$, we have that the entire enumeration of $\mathcal{A}$ can be performed in $O(n^{h+r+3})$.

Given the set $\mathcal{A}$ it is now possible to build $\mathcal{L}^*$. Start by enumerating k from 1 to r. Next, enumerate all possible ground sets $T \in \mathcal{A}$. Third, enumerate all possible subsets $\{T_1, \ldots, T_k\} \subseteq \mathcal{A}$. If $\hat{T} = \{T_1, \ldots, T_k, T\}$ defines a k-domino, and in addition, its weight satisfies the bound prescribed by Lemma 12 store it in $\mathcal{L}^*$. Otherwise, discard $\hat{T}$ and keep iterating.

Let $f(n, k)$ be the time required to test if $\{T_1, \ldots, T_k, T\}$ defines a k-domino. For each $k = 1, \ldots, h$ this second part of the algorithm requires $|\mathcal{A}|\binom{|\mathcal{A}|}{k} f(n, k)$ iterations. Since $|\mathcal{A}| = O(n^{h+r+2})$, this is equal to $O(n^{(h+r+2)(k+1)} f(n, k))$. Thus, after enumerating over all k the running time will be $O(rn^{(h+r+2)(r+1)} f(n, r))$.

Finally, note that $f(n, k) \leq O(n2^k)$, since to test if $\{T_1, \ldots, T_k, T\}$ defines a k-domino, it suffices to (a) check if $T_1, \ldots, T_k \subseteq T$, taking $O(n)$ time, and (b) check all possible subsets of $\{T_1, \ldots, T_k\}$ to see if they define the appropriate cut, taking $O(n2^k)$.

Thus we conclude that the total running time required to generate $\mathcal{L}^*$ is bounded by $n^{h+r+3} + r2^r n^{(h+r+2)(r+1)+1}$, which is $O(n^{r^2+(h+3)(r+1)+1})$.

Note that we can discard from $\mathcal{L}^*$ any 1-dominoes which are not super-connected as in Lemma 7, and that this does not affect the overall running time complexity of the procedure. □

5.2 Step 2: Putting it all together

Before actually getting to the main separation theorem, it is necessary to establish some basic graph theoretic results.

Proposition 14. *Consider a planar graph G. Let S_1, S_2 be two eulerian subgraphs of G such that $S_1 \cup S_2 = E(G)$. Further, assume that G does not contain more than $h-1$ edge disjoint cycles. Then, the number of nodes having odd degree in the subgraph induced by $S_1 \setminus S_2$ is bounded by $8h-4$.*

Proof: Let $F(G)$ represent the faces of graph G. Define two faces of G as being *adjacent* if their respective frontiers share a common edge. First, observe that $|F(G)| < 4h$. In fact, if $|F(G)| \geq 4h$, from the four-color theorem (Appel and Haken [1]) it would follow that there exists a set of h non-adjacent faces. This in turn would imply that there exist h edge disjoint cycles in $E(G)$, obtained by taking the frontier of the faces, which contradicts our problem hypotheses.

From Euler's formula we know that if $K(G)$ is the set of connected components in G, then $|E(G)| = |V(G)| + |F(G)| - 1 - |K(G)|$. Thus, since $|K(G)| \geq 1$, it follows that $|E(G)| < |V(G)| + 4h - 2$. For each $v \in V(G)$ let $d(v)$ represent the degree of v in G. Observe that because $E(G)$ is the union of two eulerian subgraphs, $d(v) \geq 2$ for all $v \in V(G)$. Define $\delta_v = 1$ if $d(v) \geq 3$, and $\delta_v = 0$ otherwise. It is clear that $2 + \delta_v \leq d(v)$ for all $v \in V(G)$. Since $\sum_{v \in V(G)} d(v) = 2|E(G)|$ it follows that

$$2|V(G)| + \sum_{v \in V(G)} \delta_v \leq \sum_{v \in V(G)} d(v) = 2|E(G)| < 2|V(G)| + 8h - 4.$$

Thus, $\sum_{v \in V(G)} \delta_v < 8h - 4$. However, every node $v \in V(G)$ having odd degree satisfies $\delta_v = 1$. Thus, the number of nodes in G having odd degree is less than $8h - 4$.

Finally, consider the subgraph induced by $S_1 \setminus S_2$. Because the number of nodes having an odd degree in this graph is equal to the number of nodes having an odd degree in G, the result follows. □

Consider a graph $G = (V, E)$ and $T \subseteq V$. A set $J \subseteq E$ is a T-Join if T is equal to the set of vertices of odd degree in the graph (V, J). For a thorough background on T-Joins, see Schrijver [22].

Proposition 15. *Consider a graph $G = (V, E)$. Assume that each edge $e \in E$ has non-negative weight x_e, and that $E = R \cup B$, where R and B are disjoint. Say that every edge in R is red, and every edge in B is blue. Consider a set $T \subseteq V$ such that $|T|$ is even. It is possible to find a minimum weight T-Join having an odd (or even) number of red edges in $O(2^{|T|} + |T|^2|V|^2 + |V|^3)$.*

Proof: Say that $F \subseteq E$ is *odd* if $|F \cap R|$ is odd; otherwise, say that F is *even*. Observe that if J is a T-Join in G, then $J = C \cup P_1 \cup P_2 \cup \ldots \cup P_k$, where each set P_i is a path with end-points in T, and set C is (a possibly empty) eulerian subgraph. Further, observe that there exists a minimum-weight odd (and even) T-Join such that: the paths P_i are pairwise edge disjoint, the set C is either empty, or an odd simple cycle, and that if P_i is odd (or even), then P_i is a minimum-weight odd (or even) path connecting its end-points.

Let C^1 be a minimum-weight odd cycle in in G'. Observe that such a cycle can be computed in $O(|V|^3)$ (see Gerards and Schrijver [11]). For each pair of distinct nodes $s, t \in T$ define P^0_{st} as a minimum-weight even path joining s and t. Likewise, define P^1_{st} as a minimum-weight odd path joining s and t. Observe that given $s, t \in T$ finding P^0_{st} and P^1_{st} can be achieved in $O(|V|^2)$ by solving a shortest path problem in an appropriate graph (see Gerards and Schrijver [11]). Computing all of the paths can thus be achieved in $O(|T|^2|V|^2)$. Define a graph G' with node set T. For each pair of distinct nodes $s, t \in T$ define an even edge in G' having end-points s, t and weight $w(P^0_{st})$, and define an odd edge in G' having end-points s, t and weight $w(P^1_{st})$. The minimum weight odd (or even) perfect-matching problem in G' can be solved by enumeration in $O(2^{|T|})$. Let $M^o \subseteq E(G')$ be the optimal even solution of this sub-problem, and let M^1 be the optimal odd solution of this sub-problem. By expanding the edges in M^o and M^1 to their corresponding paths, and iteratively removing pairs of repeated edges until each edge appears at most once in the resulting subgraphs, it is not difficult to see that we obtain T-Joins J^o and J^1 of even and odd parity. It follows that a minimum weight even T-Join is given either by J^o or $J^1 \Delta C^1$, and that a minimum weight odd T-Join is given either by J^1 or $J^o \Delta C^1$. □

Consider a graph $G = (V, E)$. Assume that the edge set E is partitioned into three subsets R, Y, B where each edge in R is labeled red, each edge in Y is labeled yellow, and each edge in B is labeled blue. In addition, assume that each edge $e \in E$ has a non-negative weight w_e. We say that a set of edges $D \subseteq E$ defines an RYB subgraph if D is eulerian and $Y \subseteq D$. Further, if $|D \cap R|$ is odd, we say that D is odd, otherwise, we say that D is even.

Proposition 16. *It is possible to find a minimum-weight odd (or even) RYB subgraph in* $O(2^{|T|} + |T|^2|V|^2 + |V|^3)$*, where* T *is the set of nodes having odd degree in the subgraph induced by* Y*.*

Proof: Let $T = \{v \in V : v \text{ is the end-point of an odd number of edges } e \in Y\}$. Observe that for any eulerian subgraph $D \subseteq E$ such that $Y \subseteq D$ we will have that $D \setminus Y$ is a T-join. Thus, our problem reduces to finding a minimum weight T-Join $J \subseteq E \setminus Y$ such that $|J \cap R|$ is odd (even). Thus the result follows from Proposition 15. □

Consider non-negative integers h, l, r and a complete set of multi-dominoes $\mathcal{L}^*$ for $\mathcal{D}(h, l, r)$. Consider a collection of large teeth $\mathcal{T}^+ \subseteq \mathcal{L}^*$ and an h-tooth association Φ^+ defined over $\mathcal{T}^+$. We say that $(\mathcal{T}, \Phi, \mathcal{R})$ defines an *extension* of $\mathcal{T}^+$ and Φ^+ if the following conditions are met:

- $\mathcal{R} = \{R_1, \ldots, R_h\}$, where $R_i \subseteq E^*$, for $i = 1, \ldots, h$.
- $\mathcal{T} \subseteq \mathcal{L}^*$ and $\mathcal{T} \setminus \mathcal{T}^+$ is a collection of 1-dominoes.
- Φ is an h-tooth association over $\mathcal{T}$, and Φ restricted to $\mathcal{T}^+$ coincides with Φ^+.
- $|\Phi^{-1}(i)|$ is odd, for $i = 1, \ldots, h$.
- $\{E(T_j : T \setminus T_j) : \Phi(\hat{T}, j) = i\}$ and $\{R_i\}$ supports a cut in G^*, for all $i = 1, \ldots, h$.

Proposition 17. *Consider non-negative integers* h, l, r *and a fractional solution* x^* *satisfying all of the subtour constraints. Assume* G^* *is planar and let* $\mathcal{L}^*$ *be a complete set of* *multi-dominoes for $\mathcal{D}(h,l,r)$. Let $\mathcal{T}^+ \subseteq \mathcal{L}^*$ be a collection of large multi-dominoes such that $|\mathcal{T}^+| \leq l$, and let Φ^+ be an h-tooth association defined over $\mathcal{T}^+$. It is possible to identify an extension $(\mathcal{T}, \Phi, \mathcal{R})$ of $\mathcal{T}^+, \Phi^+$ minimizing $s^* = \sum_{i=1}^{h} x^*(R_i) + \sum_{\hat{T}\in\mathcal{T}} w(\hat{T}) - h$ in $O(n^3)$, when h, l, r are treated as constants.*

Proof: Since G^* is a planar graph, it suffices to construct $\Phi, \mathcal{T}$ and $\mathcal{R}$ satisfying the conditions of Lemma 10. That is we must satisfy,

(i) $\mathcal{T}^+ \subseteq \mathcal{T}$ and every $\hat{T} \in \mathcal{T} \setminus \mathcal{T}^+$ is a 1-domino.

(ii) Φ restricted to $\mathcal{T}^+$ is equal to Φ^+, and $|\Phi^{-1}(i)|$ is odd, for all $i = 1, \ldots, h$.

(iii) $\{\overline{E^*(T_j : T \setminus T_j)} : \Phi(\hat{T}, j) = i\}$ and $\{\bar{R}_i\}$ support an eulerian subgraph in $\bar{G}^*$ for all $i = 1, \ldots, h$.

(iv) $s^* = \sum_{i=1}^{h} x^*(\bar{R}_i) + \sum_{\hat{T}\in\mathcal{T}} w(\hat{T}) - h$ is of minimum value.

Observe that this can be broken down into h independent problems. For each $i \in 1, \ldots, h$ determine a collection of 1-dominoes $\mathcal{T}^i \subseteq \mathcal{L}^*$ and a set $\bar{R}_i \subseteq \bar{E}$ such that,

(a) $|\mathcal{T}^i| + |\{(\hat{T}, j) : \hat{T} \in \mathcal{T}^+ \text{ and } \Phi^+(\hat{T}, j) = i\}|$ is odd.

(b) $\{\overline{E^*(T_j : T \setminus T_j)} : \hat{T} \in \mathcal{T}^+ \text{ and } \Phi^+(\hat{T}, j) = i\}$ and $\{E^*(T_1 : T \setminus T_1) : \hat{T} \in \mathcal{T}^i\}$ and $\{\bar{R}_i\}$ support an eulerian subgraph in $\bar{G}^*$.

(c) $s_i^* = x^*(\bar{R}_i) + \sum_{\hat{T}\in\mathcal{T}^i} w(\hat{T})$ is of minimum value.

In fact, if we solve each of these h problems, it is just a matter of defining $\mathcal{T} = \mathcal{T}^+ \cup \mathcal{T}^1 \cup \mathcal{T}^2 \cup \ldots \cup \mathcal{T}^h$, and $\Phi(\hat{T}, j) = \Phi^+(\hat{T}, j)$ if $\hat{T} \in \mathcal{T}^+$ and $\Phi(\hat{T}, 1) = i$ for each $\hat{T}$ in $\mathcal{T}^i$. If we do so, it is easy to see that $s^* = s_1^* + \ldots s_h^* - h + \sum_{\hat{T}\in\mathcal{T}^+} w(\hat{T})$ and that (i)-(iv) will be satisfied.

Consider any tooth $\hat{T} \in \mathcal{T}^+$ and $j \in \{1, \ldots, \kappa(\hat{T})\}$. Let $S_1 = \overline{\delta(T_j)}$ and $S_2 = \overline{\delta(T)}$. Observe that S_1 and S_2 define eulerian subgraphs in $\bar{G}^*$. Further, since $S_1 \cup S_2 = \overline{\delta(T)} \cup \overline{E(T_j : T \setminus T_j)}$, and since $x^*(\delta(T)) + x^*(E(T_j : T \setminus T_j)) < h + r + 2$ (see Lemma 12), it follows that $S_1 \cup S_2$ cannot contain more than $(h + r + 2)/2$ edge disjoint cycles (due to the subtour constraints). Thus, from Proposition 14 we conclude that the number of nodes having odd degree in $\overline{E(T_j : T \setminus T_j)}$ is less than or equal to $4(h + r + 1)$.

Now consider the multigraph G' obtained from node set $V(\bar{G}^*)$ and edges obtained from the union (allowing parallel edges) of the sets $\overline{E(T_j : T \setminus T_j)}$ such that $\hat{T} \in \mathcal{T}^+$ and $\Phi(\hat{T}, j) = i$. Let $O(G')$ represent all of the nodes having odd degree in G'. Observe that $|O(G')| \leq 4lr(h + r + 1)$. In fact, we know there are at most l teeth in $\mathcal{T}^+$, and we know that each tooth $\hat{T} \in \mathcal{T}^+$ is such that $\kappa(\hat{T}) \leq r$. Thus, there can be at most lr pairs $(\hat{T}, j)$ such that $\Phi(\hat{T}, j) = i$ with $\hat{T} \in \mathcal{T}^+$ and $j \in \{1, \ldots, \kappa(\hat{T})\}$. The bound follows from the fact that each set $\overline{E(T_j : T \setminus T_j)}$ can contribute at most $4(h + r + 1)$ odd-degree nodes to G'.

Iteratively remove from G' pairs of parallel edges until no more such pairs remain. Let Y_i be the edge set remaining (e.g., Y_i will be the set of edges appearing in an odd number of sets $\overline{E^*(T_j : T \setminus T_j)}$ with $\hat{T} \in \mathcal{T}^+$ and $\Phi^+(\hat{T}, j) = i$). Observe that each time we remove a pair of parallel edges the number of odd-degree nodes in G' does not change. Thus, the number of odd-degree nodes in the subgraph induced by Y is not more than $4lr(h + r + 1)$, or simply $O(lrh + lr^2)$.

Define another multi-graph, named M_i, having node set $V(\bar{G}^*)$. Add edge set Y_i to G'', label each of these edges as *yellow*, and assign to each a weight of zero. For each edge $e \in \bar{E}$

add an edge e' to M_i having the same end-points (call this set of edges B). Label each of these edges *blue*, and assign to them a weight equal to x^*_e. Finally, for each each 1-domino in $\mathcal{L}^*$, identify the end-nodes s and t corresponding to its domino-paths, and add an s-t edge labeled *red* to M_i with weight equal that of the domino (call this edge set R).

Consider a set of 1-dominoes $\mathcal{T}^i \subseteq \mathcal{L}^*$ and a set of edges $\bar{R}_i \subseteq \bar{E}$ satisfying (a) and (b). Let R' be the set of red edges in M_i corresponding to the 1-dominoes in $\mathcal{T}^i$, and let B' be the set of blue edges in M_i corresponding to edges in $\bar{R}_i$. Observe that $D = B' \cup Y \cup R'$ defines an RYB subgraph of M_i, as defined in Proposition 16, and that the weight of D equals exactly $x^*(\bar{R}_i) + \sum_{\hat{T} \in \mathcal{T}^i} w(\hat{T})$.

Likewise, consider an RYB subgraph $D \subseteq E(M_i)$. If $|\{\hat{T} \in \mathcal{T}^+ : \Phi(\hat{T}, j) = i, \text{ for some } j \in \{1, \ldots, \kappa(\hat{T})\}\}|$ is even, assume D is odd. Otherwise, assume D is even. Let $\mathcal{T}^i \subseteq \mathcal{L}^*$ be the 1-dominoes associated to red edges in D, and let $\bar{R}_i \subseteq \bar{E}$ correspond to the blue edges in D. Observe that $\mathcal{T}^i$ and $\bar{R}_i$ satisfy conditions (a) and (b). Further, the weight of D is equal to exactly, $x^*(\bar{R}_i) + \sum_{\hat{T} \in \mathcal{T}^i} w(\hat{T})$.

Thus, we can see that the problem of identifying the sets $\mathcal{T}^i$ and $\bar{R}_i$ such that s^*_i is minimized, reduces to the problem of finding a minimum weight RYB graph (of the appropriate parity) in M_i.

The running time of this procedure will be determined by the amount of time required to find h minimum-weight RYB subgraphs (one for each graph M_i) of the appropriate parity. From Proposition 16, this will equal $O(h2^{lrh+lr^2} + h(lrh + lr^2)^2|\bar{V}|^2 + |V|^3)$. □

We are now ready for our main result, which we prove by means of an algorithm.

Theorem 18. *Consider x^* such that all of the subtour constraints are satisfied, and such that the support graph G^* is planar. It is possible to identify a maximally violated constraint in $\mathcal{D}(h, l, r)$ in $O(n^{lr^2+l(h+3)(r+1)+l+3})$.*

Proof: The algorithm works by iterating over three loops. First, we enumerate all collections $\mathcal{T}^+ \subseteq \mathcal{L}^*$ such that $|\mathcal{T}^+| = l$. Observe that there are $\binom{|\mathcal{L}^*|}{l}$ such sets, that is $O(n^{l(r^2+(h+3)(r+1)+1)})$. Second, for each of these sets $\mathcal{T}^+$ we enumerate all possible h-tooth associations Φ^+ defined on $\mathcal{T}^+$. Given $\mathcal{T}^+$, observe that there are at most h^r such associations. Third, for each of these pairs $\mathcal{T}^+, \Phi^+$ we identify an h-parity extension $(\mathcal{T}, \Phi, \mathcal{R})$ of $\mathcal{T}^+, \Phi^+$ minimizing the quantity $s^* = \sum_{i=1}^{h} x^*(R_i) + \sum_{\hat{T} \in \mathcal{T}} w(\hat{T}) - h$. This last step requires $O(n^3)$. As we have seen in Lemma 6, each of these h-parity extensions can be used to obtain an h-parity constraint with slack s^*. Thus, among all of the extensions generated, we keep the one with smallest value s^*. Note that this h-parity constraint will be optimal (e.g., most violated). In fact, every optimal h-parity constraint must be an extension of some pair $\mathcal{T}^+, \Phi^+$, thus every possible h-parity constraint will be indirectly considered by the algorithm. Putting everything together we get an algorithm that runs in $O(n^{l(r^2+(h+3)(r+1)+1)} \cdot h^r \cdot n^3) = O(n^{lr^2+l(h+3)(r+1)+l+3})$. □

Corollary 19. *Consider x^* such that all of the subtour constraints are satisfied, and such that the support graph G^* is planar. It is possible to separate a super-class of clique-tree inequalities having h handles in $O(n^{h^4})$.*

Proof: Observe that every clique-tree on h handles has at most $h-1$ large teeth. Further, each large tooth can intersect at most h handles. Thus, considering $l = r = h$ in Theorem 18 we get the desired result. □

Note the similarity of this result with the following theorem proven by Carr [5].

Theorem 20 (Carr 1997)**.** *Consider positive integers h, t and a fractional solution x^* satisfying all of the subtour constraints. It is possible to separate the class of bipartition inequalities having h handles and t teeth in polynomial time* □

Here Carr fixes both the number of handles and the number of teeth, but allows non-planar graphs. An interesting observation is that in both cases it is not strictly necessary to specify a bound on the number of teeth having three or more halves. This is because of the observation following Lemma 5 that the number of teeth having more than three halves is naturally bounded by the number of handles. The proof of Carr's theorem follows a scheme similar to that of Theorem 18. Instead of enumerating h-tooth associations, Carr enumerates what he calls "backbones" which, essentially, are the same thing. It seems likely that Carr's proof could be easily extended to separate GDP inequalities by taking into account the observations made in this article, with the restriction that the number of 1-dominoes needs to be fixed in the case of non-planar graphs.

The algorithm we have presented is very slow for practical separation purposes. By carefully observing the proof of Theorem 18, it is possible to see that the bottleneck of the separation algorithm lies in enumerating all of the candidate sets of large teeth. In order to improve upon this key step it is necessary to go back to Proposition 13 and reduce the size of the list of candidate teeth $\mathcal{L}^*$. It seems likely that every k-domino $\hat{T} = \{T_1, \ldots, T_k, T\}$ participating in a violated h-parity constraint satisfies $x(\delta(T_j)) < 4$. The reasoning for this is that halves of a tooth in a violated inequality should be connected, and so should their complementation. Proving this likely requires using an inductive condition whereby it is assumed that before separating h-parity constraints all r-parity constraints with $r < h$ have been solved. The benefit of proving this would lie in that instead of enumerating all possible subsets of $\mathcal{A}$ having size k, one could instead enumerate subsets of $\mathcal{B} = \{B \subsetneq V : x(\delta(x)) < 4\}$ — which is considerably smaller. Another possible speed-up might be to improve upon the algorithm of Nagamochi et al. [21] by taking into account planarity and the subtour constraints. Perhaps one alternative to using the Nagamochi et al. algorithm would be to design an algorithm which works by solving shortest path problems, such as the one used by Letchford [17] to enumerate 1-dominoes. An interesting way of doing this might consist in generalizing (if possible) Lemma 7 to show that there exist maximally violated h-parity constraints where every k-domino $\hat{T} = \{T_1, \ldots, T_k, T\}$ is such that $\delta(T), \delta(T_1), \ldots, \delta(T_k), \delta(T \setminus T_1), \ldots, \delta(T \setminus T_k)$ are all minimal cuts. This might allow for a different separation approach which would work by solving shortest path problems instead of enumerating partial tooth-handle associations and then completing them.

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